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NEW SYMPLECTIC 6-MANIFOLDS VIA COISOTROPIC LUTTINGER SURGERY

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Abstract. In this paper, we study the coisotropic Luttinger surgery on several families of non-simply connected symplectic 6-manifolds. First, we show that the appropriate number of such surgeries on some of these symplectic 6-manifolds produce the simply connected symplectic and non-Kähler symplectic 6-manifolds. Next, using *coisotropic Luttinger surgery* along $\mathbb{T}^2 \times \mathbb{T}^2$, we show that for each finitely presented group G, there exists a family of symplectic 6-manifolds with fundamental group G. We also produce a variety of interesting symplectic 6-manifolds via the coisotropic Luttinger surgery on symplectic 6-manifolds such as $\Sigma_g \times \Sigma_g \times \Sigma_g$ for any $g \geq 2$, and on symplectic 6-manifold $M = (\Sigma_2 \times \mathbb{T}^2 \times \mathbb{T}^2)\#_{\Sigma_2 \times \mathbb{T}^2}((\mathbb{T}^2 \times \mathbb{S}^2 \# 4\overline{\mathbb{CP}}^2) \times \mathbb{T}^2)$, which is obtained via symplectic connected sum.

Keywords: symplectic 6-manifolds, Luttinger surgery, fundamental group.

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1 Introduction

This paper is a belated sequel to (Akhmedov, 2014) in which the author constructed the examples of simply connected symplectic Calabi-Yau 6-manifolds using the coisotropic Luttinger surgery along $\mathbb{T}^2 \times \mathbb{T}^2$. In the same paper, the author also obtained non-Kähler symplectic Calabi-Yau 6-manifolds with $b_1 = 1$ using the same surgery technique and also the symplectic connected sum operation. In this paper, we construct many new examples of simply connected symplectic 6-manifolds using coisotropic Luttinger surgery. Some of these examples presented in this article were mentioned in Akhmedov (2014), and the author promised to bring more details in future articles. The second goal of this article is to prove that for any finitely presented group G, there exists a family of symplectic 6-manifolds with $\pi_1 = G$ that can be obtained via *coisotropic Luttinger surgery* along $\mathbb{T}^2 \times \mathbb{T}^2$. This provides a new proof of the classical result in Gompf (1995), which was proved using the *symplectic connected sum* operation.

Throughout this paper $\mathbb{CP}^2$ denote the complex projective plane and $\overline{\mathbb{CP}}^2$ is the complex-projective plane with the reversed orientation. Let $X(g,n)$, $Y(g,n)$, and $Z(g,n)$ denote n fold symplectic fiber sum of the total spaces of three well known genus g hyperelliptic Lefschetz fibrations along a regular fiber Σ_g, given by the monodromies $(a_1a_2\cdots {a_{2g+1}}^2\cdots a_2a_1)^2 = 1$, $(a_1\cdots a_{2g+1})^{2g+2} = 1$, and $(a_1\cdots a_{2g})^{4g+2} = 1$ respectively, in the mapping class group M_g of the genus g surface with no punctures.

Our main results are the following four theorems.

Theorem 1. *There exist symplectic 6-manifolds with the fundamental groups*

(i) 1,

(ii) $\mathbb{Z}_p \times \mathbb{Z}_q$, *and* $\mathbb{Z} \times \mathbb{Z}_q$ *for any* $p \geq 2$ *and* $q \geq 1$

that can be obtained from $X(g,n) \times \mathbb{T}^2$, $Y(g,n) \times \mathbb{T}^2$ and $Z(g,n) \times \mathbb{T}^2$ by coisotropic Luttinger surgeries along $\mathbb{T}^2 \times \mathbb{T}^2$ for any $n \geq 2$ and $g \geq 2$.

Notice that the Lefschetz fibrations on $X(g,n)$, $Y(g,n)$ and $Z(g,n)$ have genus $g \geq 2$. We may view the above theorem as a generalization of the results of Akhmedov (2014), where the case $g = 1$ was considered, and symplectic Calabi-Yau 6-manifolds were constructed.

Our next theorem is a six-dimensional versions of the construction of symplectic 4-manifolds in Akhmedov (2008), Akhmedov & Park (2008), Akhmedov et al. (2008), Akhmedov & Park (2010), Akhmedov & Saglam (2015), Akhmedov & Ozbagci (2017) via Luttinger surgery (see Akhmedov (2008), Akhmedov & Park (2008), Akhmedov et al. (2008), Akhmedov & Park (2010), Akhmedov & Saglam (2015), Akhmedov & Ozbagci (2017) for motivation and details).

Theorem 2. *There exist symplectic 6-manifolds with the following fundamental groups*

(i) 1,

(ii) $\mathbb{Z}_p \times \mathbb{Z}_{q_1} \cdots \times \mathbb{Z}_{q_5}$, *and* $\mathbb{Z}^l \times \mathbb{Z}_{q_1} \times \cdots \mathbb{Z}_{q_{6-l}}$ *for any* $p \geq 2$, $q_i \geq 1$, *and* $1 \leq l \leq 5$

that can be obtained from the symplectic connected sum manifold $M = (\Sigma_2 \times \mathbb{T}^2 \times \mathbb{T}^2) \#_{\Sigma_2 \times \mathbb{T}^2} ((\mathbb{T}^2 \times \mathbb{S}^2 \# 4\overline{\mathbb{CP}}^2) \times \mathbb{T}^2)$ by coisotropic Luttinger surgeries.

Our next set of symplectic 6-manifolds, given in Theorems 3, are obtained by performing $1 \leq k \leq 6g$ coisotropic Luttinger surgeries on $\Sigma_g \times \Sigma_g \times \Sigma_g$ along $\Sigma_g \times \mathbb{T}^2$, where $g \geq 2$. These 6-dimensional symplectic 6-manifolds are the analogues of symplectic 4-manifolds constructed in Akhmedov (2008), Fintushel et al. (2007), Akhmedov & Park (2010).

Theorem 3. *There exist symplectic 6-manifolds with following the first homology groups in integer coefficients*

(i) 0,

(ii) $\mathbb{Z}_p \times \mathbb{Z}_{q_1} \cdots \times \mathbb{Z}_{q_{6g-1}}$, *and* $\mathbb{Z}^k \times \mathbb{Z}_{q_1} \times \cdots \mathbb{Z}_{q_{6g-k}}$ *for any* $p \geq 2$, $q_i \geq 1$, *and* $1 \leq k \leq 6g-1$

that can be obtained from $\Sigma_g \times \Sigma_g \times \Sigma_g$ by coisotropic Luttinger surgeries along $\Sigma_g \times \mathbb{T}^2$.

The following theorem is more general, and allows us to handle the case of an arbitrary finitely presented group as the fundamental group.

Theorem 4. *There exist symplectic 6-manifolds with the fundamental groups*

(i) 1,

(ii) Any finitely presented non-trivial group G given by a presentation $\langle x_1, \ldots, x_k \mid l_1, \ldots l_m \rangle$.

that can be obtained from $X(g,n) \times \Sigma_k$, $Y(g,n) \times \Sigma_k$ and $Z(g,n) \times \Sigma_k$ via coisotropic Luttinger surgeries along $\mathbb{T}^2 \times \mathbb{T}^2$ for any $n \geq 2$, and $g \geq k \geq 2$ and $g \geq k + m \geq 2$, respectively.

Our paper is organized as follows. Section 2 contains a brief review of the coisotropic Luttinger surgery. In Section 3, we collect symplectic building blocks that are needed in our construction of symplectic 6-manifolds. In Section 4, we construct symplectic 6-manifolds via coisotropic Luttinger surgery and present proofs of our main Theorems 1, 4, 2, and 3 in the given order. While the motivation of this paper is not to construct symplectic Calabi-Yau 6-manifolds, some of our bulding blocks can be used to be obtain new symplectic Calabi-Yau 6-manifolds in dimension 6 (see the author work in Akhmedov (2014)). The author will address this case in a separate article to follow.

2 Coisotropic Luttinger Surgery

In this section we will briefly review the coisotropic Luttinger surgery in dimension six and recall some basic facts about it. For the details on the coisotropic Luttinger surgery, we refer the reader to Ho (2011), Baldridge & Kirk (2013). The coisotropic Luttinger surgery has been effective tool recently for constructing symplectic Calabi-Yau 6-manifolds (Akhmedov, 2014), (Baldridge & Kirk, 2013). In this paper, we extend the effectiveness of the coisotropic Luttinger surgery further by constructing the symplectic 6-manifolds with an arbitrary finitely presented group as the fundamental group and other interesting examples. We refer the reader to Akhmedov & Zhang (2015), Akhmedov & Ozbagci (2017), Akhmedov & Saglam (2015), where the the same problem addressed in dimension 4 using *Luttinger surgery.*

Definition 1. *Let X be a closed symplectic 6-manifold with a symplectic form ω. Suppose that Σ_g is a closed 2 dimensional symplectic submanifold of X and there exist a symplectic embedding $j : D^2 \times \mathbb{T}^2 \times \Sigma_g \hookrightarrow X$ so that the submanifolds parallel to $\Lambda = \{0,0\} \times \mathbb{T}^2 \times \Sigma_g$ are all coisotropic with respect to ω. Given a simple loop λ on $\{0,0\} \times \mathbb{T}^2 \times pt$, let λ' be a simple loop on $\partial(\nu\Lambda)$ that is parallel to λ under the coisotropic framing. For any integer m, the $(\Lambda, \lambda, 1/m)$* coisotropic Luttinger surgery *on X will be $X_{\Lambda,\lambda}(1/m) = (X \setminus \nu(\Lambda)) \cup_\phi (\mathbb{T}^2 \times \Sigma_g \times D^2)$, the $1/m$ surgery on Λ with respect to λ under the coisotropic framing. Here $\phi : \mathbb{T}^2 \times \Sigma_g \times \partial D^2 \to \partial(X \setminus \nu(\Lambda))$ denotes a gluing map satisfying $\phi([\partial D^2]) = m[\lambda'] + [\mu_\Lambda]$ in $H_1(\partial(X \setminus \nu(\Lambda))$, where μ_Λ is a meridian of Λ.*

It can be shown that $X_{\Lambda,\lambda}(1/m)$ possesses a symplectic form that restricts to the original symplectic form ω on $X \setminus \nu\Lambda$. The following Lemma is an easy consequence of the the Seifert-Van Kampen's Theorem.

Lemma 1. *Let $X_{\Lambda,\lambda}(1/m)$ is obtained from X by $1/m$ coisotropic Luttinger surgery along the submanifold $\Lambda = \mathbb{T}^2 \times \Sigma_g$ of X, then the Euler characteristic is unchanged, $e(X) = e(X_{\Lambda,\lambda}(1/m))$. The fundamental group of $X_{\Lambda,\lambda}(1/m)$ is the quotient of $\pi_1(X \setminus (\mathbb{T}^2 \times \Sigma_2 \times D^2))$ by the normal subgroup generated by a homotopy class of the circle $\phi(pt \times \partial D^2)$. Thus, $\pi_1(X_{\Lambda,\lambda}(1/m)) = \pi_1(X \setminus \Lambda)/N(\mu_\Lambda \lambda'^m)$.*

3 Symplectic Building Blocks

In this section, we review the families of Lefschetz fibrations $X(g,n)$, $Y(g,n)$ and $Z(g,n)$, and Matsumoto's genus two fibration mentioned in the statments of Theorems 1, 4 and 2 .

3.1 Three familes of hyperelliptic fibrations

Let $\alpha_1, \alpha_2, \cdots, \alpha_{2g}, \alpha_{2g+1}$ denote the collection of simple closed curves given in Figure 1, and c_i denote the right handed Dehn twists t_{α_i} along the curve α_i. It is well-known that the following relations hold in the mapping class group M_g:

$$\begin{aligned}
&\Gamma_1(g) = (c_1 c_2 \cdots c_{2g-1} c_{2g} {c_{2g+1}}^2 c_{2g} c_{2g-1} \cdots c_2 c_1)^2 = 1, \\
&\Gamma_2(g) = (c_1 c_2 \cdots c_{2g} c_{2g+1})^{2g+2} = 1, \\
&\Gamma_3(g) = (c_1 c_2 \cdots c_{2g-1} c_{2g})^{2(2g+1)} = 1.
\end{aligned} \tag{1}$$

For the first monodromy relation given above, the corresponding genus g Lefschetz fibrations over $\mathbb{S}^2$ has total space $X(g,1) = \mathbb{CP}^2 \# (4g+5)\overline{\mathbb{CP}}^2$, the complex projective plane blown up at $4g+5$ points. In the case of second and third relations, the total spaces of the corresponding genus g Lefschetz fibrations over $\mathbb{S}^2$ are also well-known families of complex surfaces. For example. $Y(2,1) = K3 \# 2\overline{\mathbb{CP}}^2$ and $Z(2,1) =$ *Horikawa surface*, respectively. Moreover, it is easy to see that the monodromy relations of the genus g fibrations on $X(g,n)$, $Y(g,n)$ and $Z(g,n)$ are given by the words $\Gamma_1(g)^n$, $\Gamma_2(g)^n$, and $\Gamma_3(g)^n$, respectively.

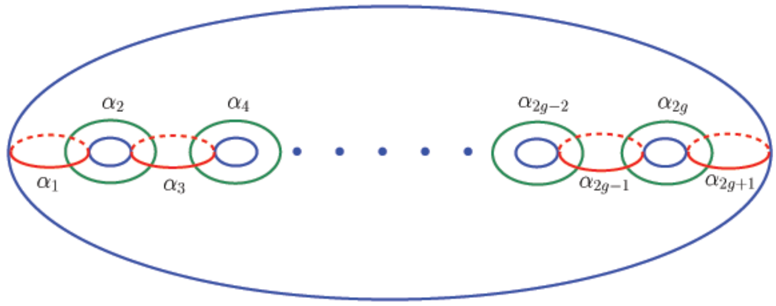

Figure 1. Vanishing Cycles of the Genus g Lefschetz Fibration on $X(g,1) = \mathbb{CP}^2\#(4g+5)\overline{\mathbb{CP}}^2$

In Fuller (1999), Fuller (2000), Akhmedov & Monden (2016), the topology of $X(g,n)$, $Y(g,n)$, and $Z(g,n)$ studied in a greater details. It is known that the complex surfaces $X(g,n)$, $Y(g,n)$, and $Z(g,n)$ have a decomposition analogous to Gompf's decomposition for elliptic surfaces. For example, $X(g,n) = W(g,n) \cup N(g,n)$, where $W(g,n)$ is diffeomorphic to the Milnor fiber of the Brieskorn homology 3-sphere $\Sigma(2, 2g+1, (4g+1)n-1)$ and $N(g,n)$ is a generalized nucleus, small submanifold with $b_2 = 2$. Similar decompositions hold for $Y(g,n)$, and $Z(g,n)$. We refer the reader to the papers Fuller (1999), Fuller (2000), Akhmedov & Monden (2016). The case $g=1$ is the well known decomposition for elliptic surfaces.

Now, we think of $X(g,2)$ as the fiber sum of two copies of $X(g,1) = \mathbb{CP}^2\#(4g+5)\overline{\mathbb{CP}}^2$ along a regular fiber Σ_g. Using the decomposition $X(g,1) = W(g,1) \cup N(g,1)$, we obtain the following decomposition of the intersection form of $X(g,2)$: $2M(g,1) \oplus H \oplus 2gH$, where H is a hyperbolic pair and $M(g,1)$ is a matrix whose entries are given by a negative definite plumbing tree given in the Figure 2. The classes that generates $M(g,1)$ all can be represented by spheres of negative self-intersection. One copy of H comes from a fiber Σ_g and a sphere section σ of self-intersection -2, i.e. from the nucleus $N(g,2)$ in $X(g,2)$. The remaining $2g$ copies of H come from $2g$ rim tori and their dual -2 spheres (see related discussion in Gompf & Stipsicz (1999), page 73)). These $12g+4$ classes ($10g+9$ spheres and $2g+1$ tori) generate $H_2(X(g,2),\mathbb{Z})$. In fact, a straightforward generalization of our argument gives the following decomposition of the intersection form of $X(g,n)$: $n(M(g,1)) \oplus H \oplus 2g(n-1)H$,

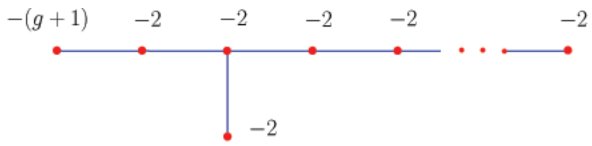

Figure 2. Plumbing Tree for the Milnor Fiber $W(g,1)$

Similarly, it is easy to obtain a decompositions of the intersection form of $Y(g,n)$ and $Z(g,n)$. We leave the details to the reader (see (Gompf & Stipsicz, 1999), (Fuller, 1999), (Fuller, 2000), (Akhmedov & Monden, 2016)).

3.2 Matsumoto fibration and their higher-genus generalizations

Yukio Matsumoto's genus two Lefschetz fibration can be conveniently described as the double branched cover of $\mathbb{S}^2 \times \mathbb{T}^2$ with the branch set being the union of two disjoint copies of $\mathbb{S}^2 \times \{\text{pt}\}$ and two disjoint copies of $\{\text{pt}\} \times \mathbb{T}^2$. The resulting branched cover has 4 singular points, corresponding to the number of intersections of the horizontal spheres and the vertical tori in the branch set. By desingularizing this manifold, we obtain the total space of Matsumoto's fibration, $M = \mathbb{T}^2 \times \mathbb{S}^2\#4\overline{\mathbb{CP}}^2$. Notice that the vertical $\mathbb{T}^2$ fibration on $\mathbb{S}^2 \times \mathbb{T}^2$ pulls back to give a fibration of $\mathbb{T}^2 \times \mathbb{S}^2\#4\overline{\mathbb{CP}}^2$ over $\mathbb{S}^2$. Since a generic fiber of the vertical fibration is the double cover of $\mathbb{T}^2$ branched over 2 points, it is a genus two surface. Matsumoto proved that Matsumoto (1996), the above fibration can be perturbed into Lefschetz one with the global monodromy given by the following word in the mapping class group M_2: $(D_1D_2D_3D_4)^2 = 1$, where D_1, D_2, D_3, and D_4 denotes the Dehn twists along the curves β_1, β_2, β_3, and β_4 shown in Figure 3.

Let us denote by Σ the regular fiber of the fibration above, and the images of the standard generators of Σ in the fundamental group of $\pi_1(M) = \mathbb{Z} \oplus \mathbb{Z}$ as a_1, b_1, a_2, and b_2. Using a homotopy long exact sequence for a Lefschetz fibration and the existence of sphere sections, we have the following identification of the fundamental group of M (Ozbagci & Stipsicz, 2000):

$$\pi_1(M) = \pi_1(\Sigma)/\langle \beta_1, \beta_2, \beta_3, \beta_4 \rangle.$$

$$\begin{aligned} \beta_1 &= b_1b_2, && (2)\\ \beta_2 &= a_1b_1{a_1}^{-1}{b_1}^{-1} = a_2b_2{a_2}^{-1}{b_2}^{-1}, && (3)\\ \beta_3 &= b_2a_2{b_2}^{-1}a_1, && (4)\\ \beta_4 &= b_2a_2a_1b_1, && (5) \end{aligned}$$

Hence $\pi_1(M) = \langle a_1, b_1, a_2, b_2 \mid b_1b_2 = [a_1, b_1] = [a_2, b_2] = b_2a_2{b_2}^{-1}a_1 = 1\rangle$.

By Van Kampen's theorem, the fundamental group of the complement of $\nu\Sigma$ in M is $\mathbb{Z} \oplus \mathbb{Z}$. This group is generated by a_1 and b_1, and the normal circle $\lambda = pt \times \partial D^2$ to Σ can be deformed using -1 sphere section of this fibration. Thus, λ is nullhomotopic in $\pi_1(\mathbb{T}^2 \times S^2 \# 4\overline{\mathbb{CP}}^2 \setminus \nu\Sigma) = \mathbb{Z} \oplus \mathbb{Z}$.

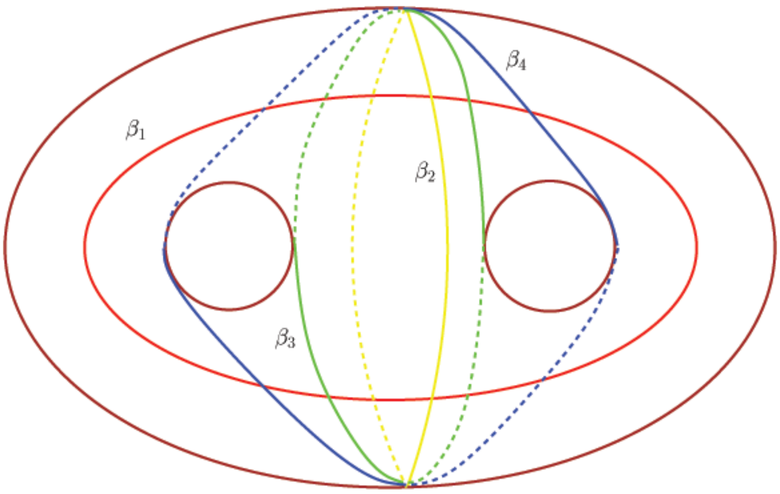

Figure 3. Dehn Twists for Matsumoto's Fibration

4 New Symplectic 6-Manifolds via Coisotropic Luttinger Surgery

In this section we prove our main theorems. For the simplicity, we will consider the case of two-fold fiber sums only, $n = 2$, in our proofs of Theorems 1 and 4. The general case is not different from this special case and the same proof applies verbatim. Throughout most of this section, Σ_g will denote a regular fiber of a hyperelliptic fibration on $X(g, n)$.

4.1 Proof of Theorem 1

Proof. Let us take two copies of $W(g) = X(g, 1) \times \mathbb{T}^2$. We endow both $W(g)$ with the product symplectic structure ω_W and form their symplectic fiber sum along the symplectic submanifolds $\Sigma_g \times \mathbb{T}^2$ and ${\Sigma_g}' \times \mathbb{T}^2$. To do this, we consider an orientation reversing gluing diffeomorphism

$\theta : \partial(\Sigma_g \times \mathbb{T}^2 \times D^2) \longrightarrow \partial(\Sigma'_g \times \mathbb{T}^2 \times D^2)$ that sends the elements of π_1 as follows:

$$\begin{aligned} a_1 = 1 &\mapsto {a'}_1 = 1, \\ b_1 = 1 &\mapsto {b'}_1 = 1, \\ &\cdots, \\ &\cdots, \\ a_g = 1 &\mapsto {a'}_g = 1, \\ b_g = 1 &\mapsto {b'}_g = 1, \\ c &\mapsto c', \\ d &\mapsto d', \\ \mu = 1 &\mapsto {\mu'}^{-1} = 1. \end{aligned}$$

Notice that the resulting symplectic 6-manifold is $X(g, 2) \times \mathbb{T}^2$. The following two essential coisotropic submanifolds are available (among many others) to perform coisotropic Luttinger surgeries in $X(g, 2) \times \mathbb{T}^2$: $\mathcal{T}_1 := (a'_1 \times c') \times (r' \times d')$ and $\mathcal{T}_2 := (b'_1 \times d'') \times (r'' \times c'')$, where r' and r'' are two disjoint the "rim" circles of $X(g, 2)$. In $X(g, 2) \times \mathbb{T}^2$, these rim circles correspond to the meridians of $\Sigma_g \times \mathbb{T}^2$. Moreover, both submanifolds $\mathcal{T}_1$ and $\mathcal{T}_2$ have dual isotropic submanifolds $\mathcal{S}_i$, which are two dimensional spheres arising from the vanishing cycles a_1 and b_1. To see this, notice that each of the above mentioned 4-dimensional torus $\mathcal{T}_i$ has a dual circle in $\Sigma_g \times \mathbb{T}^3$ intersecting $\mathcal{T}_i$ at a point. The dual spheres $\mathcal{S}_1$ and $\mathcal{S}_2$ are obtained by contracting the circles a_1 and b_1 on both sides. Since $X(g, 2)$ is simply connected and has a sphere section, a_1 and b_1 are both null-homotopic in the fundamental group of complement. Thus, we can use two vanishing disks for a_1 and b_1 to construct the spheres $\mathcal{S}_1$ and $\mathcal{S}_2$. Furthemore, the meridional circles of $\mathcal{T}_i$ lies on $\mathcal{S}_i$, thus it null-homotopic in the fundamental group of complement of $\pi_1(X(g, 2) \times \mathbb{T}^2 \setminus (\nu(\mathcal{T}_1) \cup \nu(\mathcal{T}_2))$.

Let $X(g, 2)_{p,q}$ be symplectic 6-manifold gotten by performing the following two coisotropic Luttinger surgeries on pairwise disjoint coisotropic submanifolds $\mathcal{T}_1$ and $\mathcal{T}_2$ in $X(g, 2) \times \mathbb{T}^2$: $(\mathcal{T}_1, c^p, \pm 1)$ and $(\mathcal{T}_2, d^q, \pm 1)$, where $p, q \geq 0$. Using the Lemma 1, we obtain the following presentation for the fundamental group of $X(g, 2)_{p,q}$:

$$c^p = \mu_1 = 1, \quad d^q = \mu_2 = 1, \quad [c, d] = 1. \tag{6}$$

The first coisotropic Luttinger surgery gives the relation $c^p = 1$ and the second surgery produces the relation $d^q = 1$ in $\pi_1(X(g, 2) \times \mathbb{T}^2) = \mathbb{Z} \times \mathbb{Z}$. Thus, we can realize any of the folowing abelian groups: $\mathbb{Z}_p \times \mathbb{Z}_q$ for $p, q \geq 1$, $\mathbb{Z} \times \mathbb{Z}_p$ for $p = 0$, $p \geq 2$. If we set $p = q = 1$, then $X(g, 2)_{1,1}$ has a trivial fundamental group. By setting $p = 1$, $q = 0$ or $p = 0$, $q = 1$, we get the symplectic 6-manifolds $X(g, 2)_{1,0}$ and $X(g, 2)_{0,1}$ with fundamental group $\mathbb{Z}$. Since the fist Betti numbers of $X(g, 2)_{1,0}$ and $X(g, 2)_{0,1}$ $b_1 = 1$ are odd, they are both non-Kähler symplectic 6-manifolds.

The constructions for the cases $Y(g, 2) \times \mathbb{T}^2$ and $Z(g, 2) \times \mathbb{T}^2$ are analogous. Again, the key idea is to use the rim tori in $Y(g, 2)$ and $Z(g, 2)$ resulting from the symplectic fiber sum. These homologically essential rim tori do not exist in $X(g, 1)$, $Y(g, 1)$ and $Z(g, 1)$, and arises via fiber sum. This concludes the proof of Theorem 1. □

Remark 1. *Our construction above can be modified to get a family of symplectic (both Kähler and non-Kähler)* 6*-manifolds starting with two copies of* $V(g, n) = X(g, n) \times \Sigma_g$ *and forming their the symplectic connected sum along the* 4*-dimensional symplectic submanifolds* $\Sigma_g \times \Sigma_g$ *using any orientation reserving gluing diffeomorphism* $\Psi : \partial(\Sigma_g \times \Sigma_g \times \mathbb{D}^2) \to \partial(\Sigma_g \times \Sigma_g \times \mathbb{D}^2)$. *In a special case, we can choose our gluing diffeomorphism* Ψ *that comes from an orientation preserving diffeomorphism* ψ *of* $\Sigma_g \times \Sigma_g$ *which interchnages the product copies* $\Sigma_g \times pt$ *and* $pt \times \Sigma_g$ *of* $\Sigma_g \times \Sigma_g$*, i.e. the gluing diffeomorphism* Ψ *that sends the elements of* π_1 *as follows:*

$$\begin{aligned} a_1 = 1 &\mapsto {c'}_1 = 1, \\ b_1 = 1 &\mapsto {d'}_1 = 1, \end{aligned}$$

$$
\begin{aligned}
&\cdots,\\
&\cdots,\\
a_g = 1 &\mapsto {c'}_g = 1,\\
b_g = 1 &\mapsto {d'}_g = 1,\\
c_1 = 1 &\mapsto {a'}_1 = 1,\\
d_1 = 1 &\mapsto {b'}_1 = 1,\\
&\cdots,\\
&\cdots,\\
c_g = 1 &\mapsto {a'}_g = 1,\\
d_g = 1 &\mapsto {b'}_g = 1,\\
\mu = 1 &\mapsto {\mu'}^{-1} = 1.
\end{aligned}
$$

In this special case, the above construction yields to the simply connected symplectic 6*-manifold* $X_{n,g,\psi}$ *and generalizes the construction given in Akhmedov (2014) where* $g = 1$ *was studied.*

Remark 2. *In the construction above, we could have chosen to perform the coisotropic Luttinger surgeries along* $\Sigma_g \times \mathbb{T}^2$ *using the coisotropic submanifolds* $\mathcal{L}_1 := \Sigma'_g \times (r' \times c)$ *and* $\mathcal{L}_2 := \Sigma''_g \times (r'' \times d)$*, where* r' *and* r'' *are the "rim" circles in* $X(g,2)$*, and* Σ'_g *and* Σ''_g *are two special fibers of the genus* g *Lefschetz fibration on* $X(g,2)$ *dual to the rim circles* r' *and* r''*. The submanifolds* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *have dual isotropic tori of the form* $T_1 = a_1''' \times d$ *and* $T_1 = a_2''' \times c$*. The fundamental group computation follows the same steps as before.*

Our next theorem can be viewed as 6-dimensional version of the 4-manifold construction given in Akhmedov & Ozbagci (2017). We refer the reader to Akhmedov & Ozbagci (2017) for more details and for related results.

4.2 Proof of Theorem 4

Proof. The proof of the first part of the theorem is similar to that of Theorem 1. First, using the rim circles r_i, ${r_i}'$ (for $i = 1, \cdots, k$) of the fiber Σ_g and the vanishing cycles a_i, b_i (for $i = 1, \cdots, k$) of the genus g Lefschetz fibration on $X(g,2)$, we construct $2k$ homologicall essential disjoint rim tori $\mathcal{R}_{2i-1} := a_i \times r_i$ and $\mathcal{R}_{2i} := b_i \times {r_i}'$, their dual Lagrangian tori $\mathcal{T}_{2i-1} := \mathcal{R}_{2i-1} + \mathcal{S}_{2i-1}$ and $\mathcal{T}_{2i} := \mathcal{R}_{2i} + \mathcal{S}_{2i}$, where the spheres $\mathcal{S}_{2i-1}$ and $\mathcal{S}_{2i}$ are obtained by contracting the circles b_i and a_i on both sides, using the vanishing disks of b_i and a_i. These spheres $\mathcal{S}_{2i-1}$ and $\mathcal{S}_{2i}$ have the self-intersection -2 in $X(g,2)$.

Next, using the cycles c_i, d_i (for $i = 1, \cdots, k$) of Σ_k, we see that the symplectic 6-manifold $X(g,2) \times \Sigma_k$ contains at least $2k$ disjoint essential coisotropic submanifolds of the form $\mathbb{T}^2 \times \mathbb{T}^2$, which are given by $\mathcal{V}_{2i-1} := \mathcal{T}_{2i-1} \times (r_i \times c_i)$ and $\mathcal{V}_{2i} := \mathcal{T}_{2i} \times ({r_i}' \times d_i)$.

Let $X(g,2)(p_1, q_1, \cdots, p_k, q_k)$ denote be symplectic 6-manifold gotten by performing the following $2k$ coisotropic Luttinger surgeries on $\mathcal{V}_i$ in $X(g,2) \times \Sigma_k$: $(\mathcal{V}_1, {c_1}^{p_1}, \pm 1)$, $(\mathcal{V}_2, {d_1}^{q_1}, \pm 1)$, $\cdots$, $(\mathcal{V}_{2k-1}, {c_k}^{p_k}, \pm 1)$, $(\mathcal{V}_{2k}, {d_k}^{q_k}, \pm 1)$, where c_i, d_i denote the standard generators of $\pi_1(X(g,2) \times \Sigma_k) = \pi_1(\Sigma_k)$ and $p_i, q_i \geq 0$ are integers.

4-tori $\mathcal{V}_{2i-1}$ and $\mathcal{V}_{2i-1}$ have a transversal isotropic 2-tori $\mathcal{U}_{2i-1} := b_i \times d_i$ and $\mathcal{U}_{2i} := a_i \times c_i$ respectively. The meridional circles μ_i of $\mathcal{V}_i$ are all null-homotopic in $\pi_1(X(g,2) \times \Sigma_k \setminus (\nu(\mathcal{V}_1) \cup \nu(\mathcal{V}_2) \cup \cdots \cup \nu(\mathcal{V}_{2k}))$ since the loops a_i and b_i are nullhomotopic. Furthermore, using the Lemma 1, we acquire the following presentation for the fundamental group of $X(g,2)(p_1, q_1, \cdots, p_k, q_k)$:

$$
\begin{aligned}
{c_1}^{p_1} &= \mu_1 = 1,\\
{d_1}^{q_1} &= \mu_2 = 1,\\
&\cdots,\\
&\cdots,
\end{aligned}
$$

$$
\begin{aligned}
{c_k}^{p_k} &= \mu_{2k-1} = 1, \\
{d_k}^{q_k} &= \mu_{2k} = 1, \\
\Pi_{j=1}^{k}[c_j, d_j] &= 1.
\end{aligned}
$$

By setting $p_1 = q_1 = \cdots = p_k = q_k = 1$, we obtain a symplectic 6-manifold $X(g,2)(1,1,\cdots,1,1)$ with trivial fundamental group.

Now, let $G = \langle x_1, \ \ldots, \ x_k \mid l_1, \ \ldots \ l_m \rangle$ be any finitely presented group with the given presentation. We start with the symplectic manifold $X(g,2) \times \Sigma_k$ and fix a collection of simple loops c_i and d_i representing the standard generators of the fundamental group of $X(g,2) \times \Sigma_k$ given as above. Next, we choose m additional curves γ_j in Σ_g representing the relations l_j (for $j = 1, \cdots, m$). Without loss of generality, we can assume that the curves γ_j are embedded. This can be achieved because of the assumption that $g \geq k + m$ (see discussion in (Akhmedov & Ozbagci, 2017), pages 10-12 and (Akhmedov & Zhang, 2015)). To clarify this point further (see (Akhmedov & Ozbagci, 2017; Akhmedov & Zhang, 2015)), note that by using a 1-handle attachments to Σ_k, we can resolve the intersection points of γ_j: make the curve γ_j to go over using the attached 1-handle, remove all intersection points, make the resulting curves γ_j' embedded in Σ_g, and start with the building block $X(g,2) \times \Sigma_g$, instead of $X(g,2) \times \Sigma_k$. To obtain the given presentation, we do need to perform extra $2(k-g)$ coisotropic Luttinger surgeries along $(\mathcal{V}_{2k+1}, c_{k+1}, \pm 1)$, $(\mathcal{V}_{2k+2}, d_{k+1}, \pm 1)$, $\cdots$, $(\mathcal{V}_{2g-1}, c_{2g-1}, \pm 1)$, $(\mathcal{V}_{2g}, d_{2g}, \pm 1)$ in $X(g,2) \times \Sigma_g$, which amount to the killing all extra $2(g-k)$ the fundamental group generators c_{k+1}, d_{k+1}, $\cdots$, c_{2k}, d_{2k} of $X(g,2) \times \Sigma_g$.

Let $X(g,2)(0, q_1, \cdots, 0, q_k)$ denote the symplectic 6-manifold obtained by performing the following $2k$ coisotropic Luttinger surgeries on $\mathcal{V}_i$ in $X(g,2) \times \Sigma_k$: $(\mathcal{V}_2, d_1, \pm 1)$, $\cdots$, $(\mathcal{V}_{2g}, d_k, \pm 1)$. By our discussion given above, the symplectic 6-manifold $X(g,2)(1, 0, \cdots 1, 0)$ has the fundamental group F_k, a free group of rank k. Finally, by performing m additional coisotropic Luttinger surgeries on submanifolds $\mathcal{L}_j := \mathcal{T}_{2j-1} \times (r_j \times \gamma_j)$ and $\mathcal{L}_{2j} := \mathcal{T}_{2j} \times ({r_j}' \times \gamma_{j+1})$, where $k+1 \leq i \leq g$, in $X(g,2) \times \Sigma_k$: $(\mathcal{L}_1, \gamma_1, \pm 1)$, $\cdots$, $(\mathcal{L}_m, \gamma_m, \pm 1)$, we introduce the needed relations l_i in order to collapse the free group F_k into G. The cases $Y(g,2) \times \Sigma_k$ and $Z(g,2) \times \Sigma_k$ treated similarly and we leave the details to the reader. This concludes the proof of the theorem.

□

4.3 Proof of Theorem 2

Proof. We will consider the symplectic 6-manifolds $\Sigma_2 \times \mathbb{T}^2 \times \mathbb{T}^2$ and $(\mathbb{T}^2 \times \mathbb{S}^2 \# 4\overline{\mathbb{CP}}^2) \times \mathbb{T}^2$, both endowed with the standard product symplectic structures. Let us denote the generators of the fundamental group of $\Sigma_2 \times \mathbb{T}^2 \times \mathbb{T}^2$ as a_1, b_1, a_2, b_2, c, d, e and f, and of the fundamental group of $(\mathbb{T}^2 \times \mathbb{S}^2 \# 4\overline{\mathbb{CP}}^2) \times \mathbb{T}^2)$ as c_1, d_1, x, and y.

We consider an orientation reversing gluing diffeomorphism $\theta' : \partial(\Sigma_2 \times \mathbb{T}^2 \times D^2) \longrightarrow \partial(\Sigma \times \mathbb{T}^2 \times D^2)$ that sends the elements of π_1 as follows:

$$
\begin{aligned}
a_1 &\mapsto c_1, \\
b_1 &\mapsto d_1, \\
a_2 &\mapsto {c_1}^{-1}, \\
b_2 &\mapsto {d_1}^{-1}, \\
c &\mapsto x, \\
d &\mapsto y, \\
\mu &\mapsto {\mu'}^{-1}.
\end{aligned}
$$

Let M denote their symplectic connected sum along the symplectic submanifolds $\Sigma_2 \times \mathbb{T}^2 \times pt$ and $\Sigma \times \mathbb{T}^2$ given by θ', where Σ is a regular fiber of Matsumoto's genus two fibration (See Section 3.2). By Seifert-Van Kampen theorem, the fundamental group of the resulting manifold M can be seen to be generated by a_1, b_1, c, d, e, and f which all commute with each other. Thus, the fundamental group of M is $\pi_1(M) = \mathbb{Z}^6$. The following six disjoint essential coisotropic submanifolds, four of the form $\mathbb{T}^2 \times \mathbb{T}^2$ and two of the form $\Sigma_2 \times \mathbb{T}^2$, are available to perform coisotropic Luttinger surgeries in M: $\mathcal{S}_1 := a_1' \times c' \times d' \times e'$, $\mathcal{S}_2 := b_1' \times c' \times d' \times e''$, $\mathcal{S}_3 := a_2' \times c' \times d' \times e'$, $\mathcal{S}_4 := a_2'' \times c' \times d' \times f'$, $\mathcal{U}_1 := \Sigma_2 \times c' \times e'$, and $\mathcal{U}_2 := \Sigma_2 \times d' \times e''$. Moreover, we have the following obvious dual isotropic two tori: $\mathcal{T}_1 := b_1' \times f'$, $\mathcal{T}_2 := a_1' \times f'$, $\mathcal{T}_3 := b_2' \times f'$, $\mathcal{T}_4 := b_2' \times e'$, $\mathcal{V}_1 := d' \times f'$, and $\mathcal{V}_2 := c' \times f'$. Using these 2-dimensional tori, we will determine the meridional circles to the six coisotropic submanifolds listed above.

Let $M(p, q_1, \cdots q_5)$ be symplectic 6-manifold gotten by performing the following six coisotropic Luttinger surgeries on pairwise disjoint coisotropic submanifolds $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, $\mathcal{S}_4$, $\mathcal{U}_1$, and $\mathcal{U}_2$ in M: $(\mathcal{S}_1, {a_1'}^p, -1)$, $(\mathcal{S}_2, {b_1'}^{q_1}, -1)$, $(\mathcal{S}_3, {e'}^{q_2}, 1)$, $(\mathcal{S}_4, {f'}^{q_3}, 1)$, $(\mathcal{U}_1, c^{q_4}, -1)$, and $(\mathcal{U}_2, d^{q_5}, -1)$, where $p, q_i \geq 0$.

Using the Lemma 1, we obtain the following presentation for the fundamental group of $M(p, q_1, \cdots q_5)$:

$$
\begin{gathered}
{a_1}^p = [{b_1}^{-1}, f^{-1}], {a_2}^{q_1} = [a_1^{-1}, f], e^{q_2} = [{b_2}^{-1}, f^{-1}], \\
f^{q_3} = [e^{-1}, b_2^{-1}], c^{q_4} = [d^{-1}, f^{-1}], d^{q_5} = [c^{-1}, e], \\
[a_1, c] = [a_1, d] = [a_1, e] = 1, [b_1, c] = [b_1, d] = [b_1, e] = 1 \\
[c, f] = [c, d] = [d, e] = 1 \\
a_1 a_2 = [a_1, b_1] = [a_2, b_2] = b_1 b_2 = 1
\end{gathered}
$$

By setting $p = q_1 = \cdots = q_5 = 1$, we obtain a symplectic 6-manifold $M(1, 1, \cdots, 1)$ with trivial fundamental group. To prove $\pi_1(M(1, 1, \cdots 1)) = 1$, it is enough to prove that $f = 1$, which in turn will imply that all other generators are trivial. Using the last set of identities, we have ${a_2}^{-1} = a_1, \cdots, {b_2}^{-1} = b_1$. Now, we rewrite the relation $f = [e^{-1}, b_2^{-1}] = [e^{-1}, b_1]$. Since $[b_1, e] = 1$, we obtain $f = 1$. To realize the fundamental groups stated as in (ii), we simply vary $p \geq 2$, $q_i \geq 1$ or set $p_1 = q_5 = \cdots q_{6-l+1} = 0$ and $q_i \geq 1$ for $1 \leq i \leq 6 - l$, respectively, in the presentation above. $\square$

4.4 Proof of Theorem 3

Proof. Let $X(g, g, g)$ be the product 6-manifold $\Sigma_g \times \Sigma_g \times \Sigma_g$ equipped with the product symplectic structure. Let us denote the standard generators of the fundamental group of $X(g, g, g)$ as a_1, $b_1, \cdots, a_g, b_g, c_1, d_1, \cdots, c_g, d_g, e_1, f_1, \cdots, e_g$, and f_g. Using the product structure and applying Künneth's formula, we compute the Euler characteristic and the Betti numbers of $X(g, g, g)$: $\chi(X(g, g, g)) = \chi(\Sigma_g)^3 = 8(1 - g)^3$, $b_1(X(g, g, g)) = 6g = b_5(X(g, g, g))$, $b_2(X(g, g, g)) = b_0(\Sigma_g) \cdot b_2(\Sigma_g \times \Sigma_g) + b_1(\Sigma_g) \cdot b_1(\Sigma_g \times \Sigma_g) + b_2(\Sigma_g) \cdot b_0(\Sigma_g \times \Sigma_g) = (4g^2 + 2) + 2g(4g) + 1 = 12g^2 + 3 = b_4(X(g, g, g))$, $b_3(X(g, g, g)) = b_0(\Sigma_g) \cdot b_3(\Sigma_g \times \Sigma_g) + b_1(\Sigma_g) \cdot b_2(\Sigma_g \times \Sigma_g) + b_2(\Sigma_g) \cdot b_1(\Sigma_g \times \Sigma_g) = 4g + 2g(4g^2 + 2) + 4g = 8g^3 + 12g$.

The following $6g$ homologically essential coisotropic submanifolds of the form $\Sigma_g \times \mathbb{T}^2$ are available, among many others, to perform coisotropic Luttinger surgeries in $X(g, g, g)$: $U_{0,i,i} := \Sigma_g \times c_i' \times e_i'$, $V_{0,i,i} := \Sigma_g \times d_i' \times e_i''$, $U_{i,0,i} := a_i' \times \Sigma_g \times f_i'$, $V_{i,0,i} := a_i'' \times \Sigma_g \times e_i'$, $U_{i,i,0} := a_i' \times d_i' \times \Sigma_g$, and $V_{i,i,0} := b_i' \times d_i'' \times \Sigma_g$. Moreover, we have the following dual isotropic two dimensional tori: $T_{0,i,i} := d_i' \times f_i'$, $\overline{T}_{0,i,i} := c_i' \times f_i'$, $T_{i,0,i} := b_i' \times e_i'$, $\overline{T}_{i,0,i} := b_i' \times f_i'$, $T_{i,i,0} := b_i' \times c_i'$, and $\overline{T}_{i,i,0} := a_i' \times c_i'$. Using these dual tori, we will easily identify the the meridional circles to the $6g$ aforementioned coisotropic submanifolds and compute the fundamental group of the resulting symplectic 6-manifold.

Let $X(q_1, \cdots, q_{6g})$ be symplectic 6-manifold obtained by performing the following $6g$ coisotropic Luttinger surgeries on the above family of pairwise disjoint coisotropic submanifolds $U_{0,i,i}$, $V_{0,i,i}$, $U_{i,0,i}$, $V_{i,0,i}$, $U_{i,0,i}$, and $V_{i,0,i}$ in $X(g, g, g)$:

$$
\begin{gathered}
(U_{0,1,1}, {c_1'}^{q_1}, \pm 1), (U_{0,2,2}, {c_2'}^{q_2}, \pm 1), \cdots, (U_{0,i,i}, {c_i'}^{q_i}, \pm 1), \cdots, (U_{0,g,g}, {c_g'}^{q_g}, \pm 1), \\
(V_{0,1,1}, {d_1'}^{q_g+1}, \pm 1), (V_{0,2,2}, {d_2'}^{q_{g+1}}, \pm 1) \cdots, (V_{0,i,i}, {d_i'}^{q_{g+i-1}}, \pm 1), \cdots, (V_{0,g,g}, {d_g'}^{q_{2g}}, \pm 1), \\
(U_{1,0,1}, {f_1'}^{q_{2g}+1}, \pm 1), (U_{2,0,2,}, {f_2'}^{q_{2g+1}}, \pm 1), \cdots, (U_{i,0,i}, {f_i'}^{q_{2g+i-1}}, \pm 1), \cdots, (U_{g,0,g}, {f_g'}^{q_{3g}}, \pm 1),
\end{gathered}
$$

$$(V_{1,0,1}, {e'_1}^{q_{3g}+1}, \pm 1), (V_{2,0,2,}, {e'_2}^{q_{2g+1}}, \pm 1) \cdots, (V_{i,0,i}, {e'_i}^{q_{3g+i-1}}, \pm 1), \cdots, (V_{g,0,g}, {e'_g}^{q_{4g}}, 1),$$
$$(U_{1,1,0}, {a'_1}^{q_{4g}+1}, \pm 1), (U_{2,2,0}, {a'_2}^{q_{4g+1}}, \pm 1), \cdots, (U_{i,i,0}, {a'_i}^{q_{4g+i-1}}, \pm 1), \cdots, (U_{g,g,0}, {a'_g}^{q_{5g}}, \pm 1),$$
$$(V_{1,1,0}, {b'_1}^{q_{5g}+1}, \pm 1), (V_{2,2,0}, {b'_2}^{q_{5g+1}}, \pm 1), \cdots (V_{i,i,0}, {b'_i}^{q_{5g+i-1}}, \pm 1), \cdots, (V_{g,g,0}, {b'_{g-1}}^{q_{6g}}, \pm 1),$$

Using the Lemma 1, we see that the following relations hold in the fundamental group of $X(q_1, \cdots, q_{6g})$:

$$
\begin{gathered}
[d_1^{-1}, f_1^{-1}] = c_1^{\mp q_1}, \quad [d_2^{-1}, f_2^{-1}] = c_2^{\mp q_2} \quad \cdots, [d_g^{-1}, f_g^{-1}] = c_g^{\mp q_g}, \\
[c_1^{-1}, f_1] = d_1^{\mp q_{g+1}}, \quad [c_2^{-1}, f_2] = d_2^{\mp q_{g+2}}, \quad \cdots, [c_g^{-1}, f_g] = d_g^{\mp q_{2g}}, \\
[e_1^{-1}, b_1^{-1}] = f_1^{\mp q_{2g+1}}, \quad [e_2^{-1}, b_2^{-1}] = f_2^{\mp q_{2g+2}}, \quad \cdots, [e_g^{-1}, b_g^{-1}] = f_g^{\mp q_{3g}}, \\
[f_1^{-1}, b_1] = e_1^{\mp q_{3g+1}}, \quad [f_2^{-1}, b_2] = e_2^{\mp q_{3g+1}}, \quad \cdots, [f_g^{-1}, b_g] = e_g^{\mp q_{4g}}, \\
[b_1^{-1}, c_1^{-1}] = a_1^{\mp q_{4g+1}}, \quad [b_2^{-1}, c_2^{-1}] = a_2^{\mp q_{4g+2}}, \quad \cdots, [b_g^{-1}, c_g^{-1}] = a_g^{\mp q_{5g}}, \\
[a_1^{-1}, c_1] = b_1^{\mp q_{5g+1}}, \quad [a_2^{-1}, c_2] = b_2^{\mp q_{5g+2}}, \quad \cdots, [a_g^{-1}, c_g] = b_g^{\mp q_{6g}},
\end{gathered}
$$

Notice that by setting $q_1 = \cdots = q_{6g} = \pm 1$, we obtain the symplectic 6-manifolds with trivial the first homology group in integer coefficients. To realize the first homology groups stated as in (ii), we simply set $q_1 \geq 2$, $q_i \geq 1$, and $q_1 = q_{6g-k+1} = \cdots = q_{6g} = 0$, $q_i \geq 2$ (for $1 \leq i \leq 6g-k$), respectively in the above presentation. $\square$

Remark 3. *Each non-trivial coisotropic Luttingery kills both the coisotroptic and it's dual isotroptic submanifolds and has no effect on third homology and the Euler characteristic. The Betti numbers of $X(q_1, \cdots, q_{6g})$ can be computed. For example, when $q_1 = \cdots = q_{6g} = \pm 1$, we compute the Euler characteristic and the Betti numbers of $X(q_1, \cdots, q_{6g})$ as follows: $\chi(X(q_1, \cdots, q_{6g})) = \chi(X(g, g, g)) = 8(1-g)^3$, $b_1(X(q_1, \cdots, q_{6g})) = b_1(X(g, g, g)) - 6g = 0 = b_5(X(q_1, \cdots, q_{6g}))$, $b_2(X(q_1, \cdots, q_{6g})) = b_2(X(g, g, g)) - 6g = 12g^2 - 6g + 3 = b_4(X(q_1, \cdots, q_{6g}))$, $b_3(X(q_1, \cdots, q_{6g})) = b_3(X(g, g, g)) = 8g^3 + 12g$.*

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